Tutorial 8

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1. **Theorem** Assume the boundary conditions are symmetric. If

$$f(x)f'(x)\Big|_{x=a}^{x=b} \leq 0$$

for all (real-valued) functions $f(x)$ satisfying the BCs, then there is no negative eigenvalue.

Proof: Let λ be an eigenvalue, $X(x)$ be its eigenfunction.(Actually, we have known that λ is real since the BCs are symmetric.) Then we have

$$-X''(x) = \lambda X(x).$$

Multiply the above equation by $X(x)$ and integrate w.r.t x, then

$$-\int_a^b X''(x)X(x)dx = \int_a^b \lambda X(x)^2$$

The L.H.S satisfies

$$-\int_a^b X''(x)X(x)dx = \int_a^b X'^2(x)dx - (X'X)|_a^b \geq 0.$$

by taking $f(x) = X(x)$. Therefore

$$\lambda \int_a^b X^2(x)dx \geq 0.$$

Hence we get $\lambda \geq 0$ since $X \not\equiv 0$.

2. (Exercise 4 on P129) Let

$$g_n(x) = \begin{cases} 1 \text{ in the interval } \left[\frac{1}{4} - \frac{1}{n^2}, \frac{1}{4} + \frac{1}{n^2}\right) & \text{for odd } n \\ 1 \text{ in the interval } \left[\frac{3}{4} - \frac{1}{n^2}, \frac{3}{4} + \frac{1}{n^2}\right) & \text{for even } n \\ 0 & \text{for all other } x. \end{cases}$$

Show that $g_n(x) \to 0$ in the L^2 sense but that $g_n(x)$ does not tend to zero in the pointwise sense.

Solution: On the one hand,

$$\int_{-\infty}^{\infty} |g_n(x) - 0|^2 dx = \begin{cases} \displaystyle\int_{\frac{1}{4}-\frac{1}{n^2}}^{\frac{1}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0 \quad n : \text{odd} \\ \displaystyle\int_{\frac{3}{4}-\frac{1}{n^2}}^{\frac{3}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0 \quad n : \text{even} \end{cases} \quad \text{as } \ n \to \infty$$

Hence $g_n(x) \to 0$ in the L^2 sense.

On the other hand, for all odd n, we have $g_n(\frac{1}{4}) = 1$ which implies that $g_n(x)$ cannot tend to zero in the pointwise sense.